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M.Sc. (Sem.- III) Examination October - 2025

Mathematics (Paper - PGMTH: 3045) Advanced Special Functions-I

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book (2) All questions are compulsory. (3) Figures to the right indicate marks of the questions. (4) Follow usual notations.	Student's Signature

Q.1. Attempt any <u>TWO</u>: **14**

(I) If n is a non negative integer and if b and c are independent of n, show that

$$ {}_3F_2\left[\begin{matrix} -n, b, c; \\ 1-b-n, 1-c-n; \end{matrix}\ x\right] = (1-x)^n\ {}_3F_2\left[\begin{matrix} \frac{-n}{2}, \frac{-n}{2}+\frac{1}{2}, 1-b-c-n; & \frac{-4x}{(1-x)2} \\ 1-b-n, \quad 1-c-n; & \end{matrix}\right] $$

(II) Show that $w = {}_pF_q$ is a solution of the Generalized Hypergeometric differential equation $\left[\theta \underset{j=1}{\overset{q}{\pi}} (\theta + b_j - 1) - z \underset{i=1}{\overset{p}{\pi}} (\theta + a_i)\right] w = 0$

(III) Obtain the contiguous function relation

$$[(1-x) + (A-B)x]\, F = (1-x)a_1\, F(a_1 +) - x \sum_{j=1}^{q} U_j\, F(b_j +),\ p = q+1$$

Q.2. Attempt any <u>TWO</u>: **14**

(I) Show that if p = q and no a_m is zero or a negative integer show that

$$\frac{1}{2\pi i}\int_B \frac{\overline{|(-s)}\,(-z)^s \underset{m=1}{\overset{p}{\pi}} \overline{|(a_m+s)}}{\underset{j=1}{\overset{q}{\pi}} \overline{|b_j+s)}}$$ where B is Barnes contour; is an analytic

function of Z in the open half plane $|\arg(z)| < \frac{\pi}{2}$.

(II) If $|z| < 1$, $|\arg(-z)| \le \pi - \delta$, $\delta > 0$ and if no a_m or b_j is zero or a negative integer then

$$\frac{1}{2\pi \mathrm{i}}\int_{B_n} \frac{(-z)^s \overline{|(-s)} \underset{m=1}{\overset{q+1}{\pi}} \overline{|(a_m+s)}}{\underset{j=1}{\overset{q}{\pi}} \overline{|(b_j+s)}}\, ds = \frac{\underset{m=1}{\overset{q+1}{\pi}} \overline{|(a_m)}}{\underset{j=1}{\overset{q}{\pi}} \overline{|(b_j)}}\ {}_{q+1}F_q\left[\begin{matrix} a_1, a_2, \ldots, a_{q+1}; \\ b_1, b_2, \ldots, b_q; \end{matrix}\ z\right]$$

(III) If n is a non-negative integer and if a, b, c are independent of n, then

prove that $\quad {}_3F_2\left[\begin{matrix} -n, \ a, \ b \ ; \\ c, 1-c+a+b-n\,; \end{matrix}\ 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}$

Q.3. **Attempt any <u>TWO</u>:** **14**

(I) State and prove the Kumer's first formula

(II) What is the concept of generating function ? For the function $G(2xt - t^2) = \sum_{n=0}^{\infty} g_n(x)t^n$, show that $g'_0(x) = 0$ *and* for $n \geq 1$, $xg'_n(x) - ng_n(x) = g'_{n-1}(x)$

(III) For integral n show that $J_n(z) = \frac{1}{\pi}\int_0^{x} Cos(n\theta - z\sin\theta)\,d\theta$

Q.4. **Attempt any <u>TWO</u>:** **14**

(I) Let the polynomials $f_n(x)$ are defined by $(1-t)^{-c}\,\Psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)t^n$ Where $\psi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \gamma_0 \neq 0$ then show that.

$$f_n(x) = \frac{(c)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k\,(c+n)_k\,\gamma_k\,x^k}{\left(\frac{c}{2}\right)_k\left(\frac{c}{2}+\frac{1}{2}\right)_k}$$

(II) If $P_n(x)$ is defined by $A(t)\,\psi(xH(t)) = \sum_{n=0}^{\infty} P_n(x)\,t^n$ with $\psi(t) = \sum_{n=0}^{\infty}\gamma_n t^n$, $A(t) = \sum_{n=0}^{\infty} a_n t^n$ $H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}$ where $\gamma_0 \neq 0$, $a_0 \neq 0$, $h_0 \neq 0$, then show that there exits sequences o numbers α_k and β_k such that for $n \geq 1$

$$xp'_n(x) - np_n(x) = -\sum_{k=0}^{n-1}\alpha_k P_{n-1-k}(x)$$

$$x\sum_{k=0}^{n-1}\beta_k P'_{n-1-k}(x)$$

in which $\frac{tA'(t)}{A(t)} = \sum_{n=0}^{\infty}\alpha_n t^{n+1}$, $\frac{tH'(t)}{H(t)} = 1 + \sum_{n=0}^{\infty}\beta_n t^{n+1}$

(III) Let the polynomials $f_n(x)$ are defined by $(1-t)^{-c}\,\psi\left(\frac{-4xt}{(1-t)^2}\right) = \sum_{n=0}^{\infty} f_n(x)t^n$ where $\psi(u) = \sum_{n=0}^{\infty}\gamma_n u^n$, $\gamma_0 \neq 0$ then show that

$$xf'_n(x) - n f_n(x) = -(c+n-1)f_{n-1}(x) - xf'_{n-1},\ n \geq 1$$

Q.5. **Attempt any <u>TWO</u>:** **14**

(I) For $t \neq 0$ and for all finite z show that $\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \Sigma_{-\infty}^{\infty} J_n(z)t^n$

(II) Show that the Neumann polynomial's $0_n(s)$ satisfy the relations $0_0(s) = s^{-1}$, $0_1(s) = s^{-2}$, $0_n(s) = 0_{n-2}(s) - 2\,0'_{n-1}(s)$, $n \geq 2$.

(III) Show that the Neumann polynomial's defined by $0_0(s) = s^{-1}$, $0_1(s) = s^{-2}$, $0_n(s) = 0_{n-2}(s) - 2\,0'_{n-1}(s)$, $n \geq 2$. are given by $O_0(s) = s^{-1}$, and $O_n(s) = \frac{n}{4}\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(n-1-k)!}{k!}\left(\frac{2}{s}\right)^{n+1-2k}; n \geq 1$
